Range, Null Space, and Orthogonality

Companion notes for the "Spaces and Transformations" slideshow. Section numbers below match the slideshow's own three worked examples; light click markers help you follow along either while presenting or while reading.

Every linear map A has a shape, and that shape is best understood through four subspaces: its **range** (what it can reach), its **null space** (what it collapses to zero), its **row space** — the range of A^T — and its **left null space** — the null space of A^T. The three examples below walk through what those subspaces look like geometrically, starting simple and building up.

Example 1: An invertible transformation

We start with

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

and watch the whole plane transform under it. Splitting the screen into a domain panel $\mathcal{U}$ and a codomain panel $\mathcal{V}$, we send a rainbow cloud of points from $\mathcal{U}$ across to $\mathcal{V}$ — the color of each point tracks its original position, so you can watch the transformation without losing track of *where things came from.*

Because A is invertible, the map is one-to-one and onto: nothing new appears in $\mathcal{V}$ that wasn't reachable, and nothing collapses. The range of A is all of $\mathcal{V}$, and the null space is just the single point at the origin — the origin is the *only* vector that maps to zero, since A never squashes any direction flat. That's the whole story for an invertible matrix: an invertible transpose has nothing new to show either, so there's no need to separately dig into components or A^T here — that richer structure only shows up once a map isn't invertible.

Example 2: A singular transformation

Now consider

$$A = \begin{bmatrix} 1 & 3 \\ 0.5 & 1.5 \end{bmatrix}$$

whose columns are linearly dependent (the second column is exactly 3 times the first). Transforming the plane collapses it down to a single line — this is the full template for a singular map, run in five stages.

Range and null space. Mapping the rainbow cloud from $\mathcal{U}$ to $\mathcal{V}$ again, every point lands somewhere on one line through the origin in $\mathcal{V}$: the range of A, spanned by $(2, 1)$. The null space is the set of points $\mathcal{U}$ that get crushed to zero — another line, spanned by $(3, -1)$.

Components. Take a point not on either of those special lines, say $(1, 1)$, which maps to $A(1, 1) = (4, 2)$ in $\mathcal{V}$. Every point can be split into a **row-space component** (along $(1, 3)$, the row space of A — equivalently the range of A^T) and a **null-space component** (along the null-space direction $(3, -1)$). Since the null-space component always maps to zero, the original point and its row-space component alone map to the *exact same place* in $\mathcal{V}$ — you can drop the null-space piece entirely and lose nothing. Push this further: fan a whole line of points out along the null-space direction, all passing through that same row-space component, and every single one of them maps to that one point in $\mathcal{V}$.

Turning around: A^T. Now look at

$$A^T = \begin{bmatrix} 1 & 0.5 \\ 3 & 1.5 \end{bmatrix}$$

by swapping the two panels' roles — $\mathcal{V}$ becomes the domain and $\mathcal{U}$ becomes the codomain. Its range is the row space of A (the same $(1, 3)$ line, now living in $\mathcal{U}$ as a *destination* rather than a component direction), and its null space is a new line in $\mathcal{V}$, spanned by $(1, -2)$ — the **left null space** of A.

Orthogonality. With the panels swapped back, each panel now holds two of the four subspaces at once, drawn in different colors: in $\mathcal{U}$, the null space and the row space; in $\mathcal{V}$, the range and the left null space. In both panels, the two lines meet at a right angle — the null space is always orthogonal to the row space, and the range is always orthogonal to the left null space.

Example 3: A rank-2 map from $\mathbb{R}^3$ to $\mathbb{R}^3$

The same four subspaces show up for a bigger matrix, deliberately chosen to be rank-2 (its columns are linearly dependent) so every subspace direction stays a clean, small integer vector:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

Since domain and codomain are both 3-D here, we skip straight to the two-panel view rather than transforming one shared grid. A single point, $(1, 1, 0)$, maps to $(2, 1, -1)$; a full rainbow cloud reveals that the range of A isn't all of $\mathcal{V}$ but a 2-D *plane*, spanned by $(1, 0, -1)$ and $(1, 1, 0)$. The null space is a line through $(1, 1, -1)$ in $\mathcal{U}$.

Turning around to A^T: the row space of A is now a plane in $\mathcal{U}$, spanned by $(1, 1, 2)$ and $(0, 1, 1)$, and the left null space is a line in $\mathcal{V}$ through $(1, -1, 1)$ — again orthogonal to the range plane, exactly as the null space is orthogonal to the row-space plane in $\mathcal{U}$. (With everything now a mix of planes and lines in 3-D, we skip the component-decomposition exercise from Example 2 — seeing a projection onto a plane cleanly needs more visual machinery than this video sets up — but every panel keeps rotating as new content appears, so the 3-D relationships stay legible rather than flattening into an ambiguous 2-D silhouette.)