Bases, Coordinates, and Matrix-Vector Multiplication

Companion notes for the "Bases and Matrices" slideshow. Section numbers below match the slideshow's own sections; each subsection corresponds to one click through the presentation, so you can follow along in either direction.

1. Coordinate systems

Clicks 1–3. We start with a vector, $(2,3)$, drawn in the plane. This notation is one we're all familiar with — but it quietly assumes a particular *basis*, or coordinate system. To make that assumption visible, we draw in the standard basis vectors e_1 and e_2 alongside it.

Clicks 4–7. Concretely, $(2,3)$ means:

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Each slot in the vector pairs with one basis vector — the first coordinate scales e_1, the second scales e_2. It's tempting to think of the coordinates *as* the vector, but the vector is really the coordinates together with the (usually implicit) basis they're written against.

Clicks 9–12. Change the basis, keep the coordinates fixed, and you get a genuinely different vector. Watching $(2,3)$ ride along as e_1 and e_2 morph through three different bases —

$$(e_1, e_2) = ((2,1),\ (4,3)) \ \rightarrow\ ((-1,1),\ (3,-2)) \ \rightarrow\ ((0.5,1),\ (1,-0.5))$$

— makes this concrete: the *coordinates* $(2,3)$ never change, but the arrow they describe moves all over the plane. We'll return to what makes one basis "nicer" to work with than another later in the course.

Clicks 12–14. Basis vectors don't have to be linearly independent. If e_1 and e_2 are chosen to point along the same line — here $(2,1)$ and $(3,1.5)$ — their span collapses to one dimension, and *no* choice of coordinates can escape that line. Sliding the first coordinate through $0 \rightarrow 4$ and then the second through $1 \rightarrow 5$ shows the vector sweeping back and forth, but always confined to the same 1-D span.

2. Matrix-vector multiplication

Clicks 16–17. This conversation about bases connects directly to matrix-vector multiplication. Take:

$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The key idea: matrix-vector multiplication is a *linear combination of the matrix's columns*, weighted by the vector's entries:

$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Clicks 18–20. Here's another way to see the same fact. On its own, the vector $(-1, 2)$ is really shorthand for a combination of the *standard* basis vectors:

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Multiplying by the matrix swaps in the matrix's *columns* in place of the standard basis vectors — column i answers "where does e_i go?":

$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Clicks 21–22. Carrying out the arithmetic:

$$-1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Importantly, this is *not* a change of coordinates — the coordinates $(-1, 2)$ stay exactly the same throughout. What changes is which vectors those coordinates are measured against.

3. Transformations

Clicks 23–24. Multiplying every vector in the plane by the same matrix is a *linear transformation* — and it's easiest to see by watching the whole grid move at once. Applying

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$$

to the plane drags the background grid lines along with it (they visibly skew, since that's the whole point), while $(-1, 2)$ and the basis vectors ride along, landing at $(-2, 5)$ — matching the arithmetic from Section 2.

Clicks 25–28. What happens when the matrix's columns are linearly dependent, as in

$$B = \begin{bmatrix} 1 & 3 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

? Applying B to the plane collapses every vector — regardless of its starting coordinates — onto a single line, the one-dimensional span of B's columns. A linearly dependent set of columns can't do anything else: there's simply no direction left to escape into.